**Evaluate the sum**

Let be real and , evaluate the sum .

**Method 1**

Let , then

Similarly let ,

From the given, , therefore

**Method 2**

Note that ,

By , cancelling the none-zero factor in parenthesis, we have , .

**Method 3**

Since

From given, we have and .

Adding together and cancelling the radicals, we have .

**Method 4**

Since

Taking natural logarithm,

,

Similarly, let ,

By

Therefore .

Advanced students may discern the connection of this problem with the ***hyperbolic function*** :

and .

**Method 5**

If we allow the use of ***complex numbers***, the following is quite a mysterious way.

Let

It is even more mysterious that by choosing suitable complex we can make are real.

Since

Therefore

By de’ Moivres theorem,

, where is an integer.

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