**Evaluate the sum**

Let $x,y$ be real and $\left(x+\sqrt{x^{2}+1}\right)\left(y+\sqrt{y^{2}+1}\right)=1$ , evaluate the sum $x+y$ .

**Method 1**

 Let $u=x+\sqrt{x^{2}+1}$ , then

 $\left(u-x\right)^{2}=x^{2}+1 ⇒ u^{2}-2ux+x^{2}=x^{2}+1 ⇒ u^{2}-2ux=1⇒x=\frac{ u^{2}-1}{2u}$

 Similarly let $v=y+\sqrt{y^{2}+1}$ , $y=\frac{ v^{2}-1}{2v}$

 From the given$\left(x+\sqrt{x^{2}+1}\right)\left(y+\sqrt{y^{2}+1}\right)=1$, $uv=1$, therefore $v=\frac{1}{u}$

 $x+y=\frac{ u^{2}-1}{2u}+\frac{ v^{2}-1}{2v}=\frac{ u^{2}-1}{2u}+\frac{ \left(\frac{1}{u}\right)^{2}-1}{2\left(\frac{1}{u}\right)}=\frac{ u^{2}-1}{2u}+\frac{ 1-u^{2}}{2u}=0$

**Method 2**

 Note that $\frac{1}{x+\sqrt{x^{2}+1}}=-x+\sqrt{x^{2}+1}$,

 $\left(x+\sqrt{x^{2}+1}\right)\left(y+\sqrt{y^{2}+1}\right)=1 ⟹y+\sqrt{y^{2}+1}=-x+\sqrt{x^{2}+1} ….(1)$

 $(1)^{2}, y^{2}+2y\sqrt{y^{2}+1}+y^{2}+1=x^{2}-2x\sqrt{x^{2}+1}+x^{2}+1$

 $2y\left(y+\sqrt{y^{2}+1}\right)=2x\left(-x+\sqrt{x^{2}+1}\right)$

 By $(1)$, cancelling the none-zero factor in parenthesis, we have $2y=-2x$ , $x+y=0$.

**Method 3**

 Since $\frac{1}{x+\sqrt{x^{2}+1}}=-x+\sqrt{x^{2}+1}, \frac{1}{y+\sqrt{y^{2}+1}}=-y+\sqrt{y^{2}+1}$

 From given, we have $y+\sqrt{y^{2}+1}=-x+\sqrt{x^{2}+1}$ and $x+\sqrt{x^{2}+1}= -y+\sqrt{y^{2}+1}$.

 Adding together and cancelling the radicals, we have $x+y=0 $.

**Method 4**

 Since $\left(x+\sqrt{x^{2}+1}\right)\left(y+\sqrt{y^{2}+1}\right)=1$

 Taking natural logarithm, $ln\left(x+\sqrt{x^{2}+1}\right)+ln\left(y+\sqrt{y^{2}+1}\right)=0 ….(1)$

 $Let u=ln\left(x+\sqrt{x^{2}+1}\right)⟹e^{u}=x+\sqrt{x^{2}+1} …(1)$

 $e^{-u}=\frac{1}{x+\sqrt{x^{2}+1}}=-x+\sqrt{x^{2}+1} …(2)$

 $\frac{\left(1\right)-(2)}{2}$, $x=\frac{e^{u}-e^{-u}}{2} …. (3)$

 Similarly, let $v=ln\left(y+\sqrt{y^{2}+1}\right)$ , $y=\frac{e^{v}-e^{-v}}{2} ….(4)$

 By $\left(1\right),$ $u+v=0, v=-u$

 Therefore $x+y=\frac{e^{u}-e^{-u}}{2}+\frac{e^{v}-e^{-v}}{2}=\frac{e^{u}-e^{-u}}{2}+\frac{e^{-v}-e^{v}}{2}=0$ .

 Advanced students may discern the connection of this problem with the ***hyperbolic function*** :

 $\sinh(x)=\frac{e^{x}-e^{-x}}{2}$ and $sinh^{-1}x=ln\left(x+\sqrt{x^{2}+1}\right)$.

 $sinh^{-1}x+sinh^{-1}y=0⟹x+y=0$

**Method 5**

If we allow the use of ***complex numbers***, the following is quite a mysterious way.

 Let $x=i\sin(α), y=i sin β$

 It is even more mysterious that by choosing suitable complex$ α,β$ we can make $x,y$ are real.

 Since $\left(x+\sqrt{x^{2}+1}\right)\left(y+\sqrt{y^{2}+1}\right)=1$

 Therefore $\left(i\sin(α)+\sqrt{-sin^{2} α+1}\right)\left(i\sin(β)+\sqrt{-sin^{2} β+1}\right)=1$

 $\left(\cos(α)+i\sin(α)\right)\left(\cos(β)+i\sin(β)\right)=1$

 By de’ Moivres theorem, $\cos(\left(α+β\right))+i\sin(\left(α+β\right)=1)$

 $∴α+β=2nπ$ , where $n$ is an integer.

$$∴ x+y=i\sin(α)+i sin β=i \sin(α)+i sin\left(2nπ-α\right)=i \sin(α)-i \sin(α)=0$$

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